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A REMARK ON FEJÉR AND MITTAG-LEFFLER THEOREMS

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Abstract. We discuss some generalizations of the classical Fejér and Mittag-Leffler theorems to the case of several complex variables with applications to the Shilov and Bergman boundaries.

1. Introduction. For a bounded domain $D \subset \mathbb{C}^N$ let $\mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{D})$) denote the space of all continuous functions $f : \overline{D} \longrightarrow \mathbb{C}$ such that $f|_D$ is holomorphic (resp. f extends holomorphically to a neighborhood of $\overline{D}$). Let $\partial_S D$ (resp. $\partial_B D$) be the *Shilov* (resp. *Bergman*) *boundary* of D, i.e. the minimal compact set $K \subset \overline{D}$ such that $\max\limits_K |f| = \max\limits_{\overline{D}} |f|$ for every $f \in \mathcal{A}(D)$ (resp. $f \in \mathcal{O}(\overline{D})$). Obviously, $\mathcal{O}(\overline{D}) \subset \mathcal{A}(D)$ and hence $\partial_B D \subset \partial_S D \subset \partial D$. Notice that, in general, $\partial_B D \subsetneq \partial_S D$, e.g. for the domain $D := \{(z,w) \in \mathbb{C}^2 : 0 < |z| < 1,\ |w| < |z|^{-\log|z|}\}$ (cf. [**6**], § 16).

The algebra $\mathcal{A}(D)$ endowed with the supremum norm is a Banach algebra. Then $\partial_S D$ coincides with the Shilov boundary of $\mathcal{A}(D)$ in the sense of uniform algebras (cf. [**7**], Chap. I, Sec. H). Moreover, the Bergman boundary $\partial_B D$ coincides with the Shilov boundary of the uniform algebra $\mathcal{B}(D)$ defined as the uniform closure in $\mathcal{A}(D)$ of $\mathcal{O}(\overline{D})|_{\overline{D}}$.

Assume that the envelope of holomorphy $\widetilde{D}$ of D is univalent. We are interested in characterizations of those domains D for which $\partial_S D = \partial_S \widetilde{D}$ (resp. $\partial_B D = \partial_B \widetilde{D}$) (cf. [**9**]).

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Remark 1.1. (a) It is well known (cf. [**8**], Remark 1.4.5(h)) that

$$\sup_{\widetilde{D}} |g| = \sup_{D} |g|, \quad g \in \mathcal{O}(\widetilde{D}). \tag{*}$$

In particular, if D is bounded, then so is $\widetilde{D}$.

(b) It is also well known that if $D \subset \mathbb{C}^N$ is Reinhardt (resp. balanced, resp. starlike) domain, then $\widetilde{D}$ is univalent and $\widetilde{D}$ is Reinhardt (resp. balanced, resp. starlike) (cf. [**8**], Remark 1.9.6(c,e,f), Corollary 1.9.18).

(c) Let $D \subset G \subset \mathbb{C}^N$ be domains such that G is a domain of holomorphy and (D, G) is a *Runge pair*, i.e. the space $\mathcal{O}(G)|_D$ is dense in $\mathcal{O}(D)$. Then the envelope of holomorphy of D is univalent and $(\widetilde{D}, G)$ is a Runge pair (cf. [**3**], see also [**8**], Proposition 3.1.22). In particular, the result applies if $G = \mathbb{C}^N$ and polynomials are dense in $\mathcal{O}(D)$.

First, let us recall some known results.

Remark 1.2. (a) Since $\mathcal{A}(\widetilde{D})|_{\overline{D}} \subset \mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}} \subset \mathcal{O}(\overline{D})$), we get $\partial_S \widetilde{D} \subset \partial_S D$ (resp. $\partial_B \widetilde{D} \subset \partial_B D$).

(b) If $\mathcal{A}(D) \subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$ (resp. $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}}$), then $\partial_S D = \partial_S \widetilde{D}$ (resp. $\partial_B D = \partial_B \widetilde{D}$).

(c) There exists a bounded Reinhardt domain $D \subset \mathbb{C}_* \times \mathbb{C}$ such that $\mathcal{A}(D) \not\subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$ (cf. [**11**], Example 6.1). On the other hand, if $D \subset \mathbb{C}^N$ is a bounded Reinhardt domain, then $\partial_S D = \partial_S \widetilde{D}$ (cf. [**11**], Corollary 6.2).

(d) If $\overline{D}$ has a neighborhood basis consisting of domains with univalent envelopes of holomorphy, then $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}}$ and hence $\partial_B D = \partial_B \widetilde{D}$.

In fact, by (b), we only need to prove that if $\overline{D} \subset G \subset \mathbb{C}^N$, where G is a bounded domain with a univalent envelope of holomorphy $\widetilde{G}$, then $\overline{\widetilde{D}} \subset \widetilde{G}$. For every $g \in \mathcal{O}(\widetilde{G})$, using (*), we get $\sup_{\widetilde{D}} |g| = \sup_D |g| = \max_{\overline{D}} |g|$. Thus $\widetilde{D} \subset \widehat{\overline{D}}_{\mathcal{O}(\widetilde{G})} \subset\subset \widetilde{G}$, where $\widehat{K}_{\mathcal{O}(\Omega)} := \{z \in \Omega : \forall_{g \in \mathcal{O}(\Omega)} : |g(z)| \leq \max_K |g|\}$ (cf. [**8**], Theorem 1.10.4).

(e) Let $\mathbb{B}(a, r)$ denote the Euclidean ball centered at $a \in \mathbb{C}^N$ with radius $r > 0$; $\mathbb{B}(r) := \mathbb{B}(0, r)$. If $D \subset \mathbb{C}^N$ is a bounded balanced (resp. starlike) domain, then $\{\overline{D} + \mathbb{B}(\varepsilon)\}_{\varepsilon > 0}$ gives a neighborhood basis of $\overline{D}$ consisting of bounded balanced (resp. starlike) domains. Hence $\partial_B D = \partial_B \widetilde{D}$.

(f) If $\partial_B D = \partial_B \widetilde{D}$ and $\mathcal{A}(D) = \mathcal{B}(D)$, then $\partial_S D = \partial_S \widetilde{D}$.

Indeed, $\partial_S D = \partial_B D = \partial_B \widetilde{D} \subset \partial_S \widetilde{D} \subset \partial_S D$.

(g) There exists a bounded Hartogs domain $D \subset \mathbb{C}^2$ with a univalent envelope of holomorphy $\widetilde{D}$ such that $\partial_S D \neq \partial_S \widetilde{D}$, $\partial_B D \neq \partial_B \widetilde{D}$, and $\mathcal{O}(\overline{D}) \not\subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$ (cf. [**9**]).

The paper is organized as follows:

— first, we prove two general theorems on polynomial approximation in balanced and starlike domains (Theorems 2.1, 3.1); these results are of independent interest;

— next, we prove the following result.

THEOREM 1.3. (a) *If D is a bounded balanced domain, then $\partial_S D = \partial_S \widetilde{D}$ and $\partial_B D = \partial_B \widetilde{D}$.*
(b) *If D is a bounded strictly starlike domain, then $\partial_S D = \partial_S \widetilde{D}$ and $\partial_B D = \partial_B \widetilde{D}$.*

Notice that (a) answers an open problem formulated in [**9**].

Recall that a bounded starlike domain $D \subset \mathbb{C}^n$ is said to be *strictly starlike with respect to the origin* if $\overline{D} \subset (1+\varepsilon)D$ for every $\varepsilon > 0$. The equality $\partial_S D = \partial_S \widetilde{D}$ for an arbitrary bounded starlike domain D seems to be an open problem.

2. Fejér theorem for holomorphic functions.

THEOREM 2.1 (Fejér theorem). *Let $D \subset \mathbb{C}^N$ be a bounded balanced domain, let $f \in \mathcal{A}(D)$, and let*

$$f(z) = \sum_{j=0}^{\infty} Q_j(z), \quad z \in D,$$

be the Taylor series development of f in D (Q_j is a homogeneous polynomial of degree j). Put

$$s_n := \sum_{j=0}^{n} Q_j, \quad \sigma_n := \frac{s_0 + \cdots + s_{n-1}}{n}, \quad n \in \mathbb{N}.$$

Then $\sigma_n \longrightarrow f$ uniformly on $\overline{D}$.

PROOF. It is known that for arbitrary $n \in \mathbb{N}$ and $z \in D$ we have

$$\sigma_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it} z) F_n(t) dt,$$

where

$$F_n(t) := \frac{1}{n} \Big(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \Big)^2$$

is the *n-th Fejér kernel*. In particular,

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) dt, \quad n \in \mathbb{N}.$$

Let $M := \max_{\overline{D}} |f|$, $\omega(\delta) := \max\{|f(e^{it}z) - f(z)| : z \in \overline{D},\ |t| \le \delta\}$, $\delta > 0$. Since the function $\overline{D} \times [-\pi, \pi] \ni (z,t) \longmapsto f(e^{it}z)$ is uniformly continuous, we have $\lim_{\delta \to 0+} \omega(\delta) = 0$. The standard reasoning gives

$$|\sigma_n(z) - f(z)| \le \frac{1}{2\pi} \int_{|t| \le \delta} |f(e^{it}z) - f(z)| F_n(t) dt$$
$$+ \frac{1}{2\pi} \int_{\delta < |t| \le \pi} |f(e^{it}z) - f(z)| F_n(t) dt \le \omega(\delta) + \frac{2M}{n \sin^2 \frac{1}{2}\delta}, \quad z \in D,\ 0 < \delta < \pi.$$

Consequently, given $\varepsilon > 0$, we first find a $\delta \in (0, \pi)$ such that $\omega(\delta) \le \frac{\varepsilon}{2}$ and next we choose an $n_0 \in \mathbb{N}$ with $\frac{2M}{n \sin^2 \frac{1}{2}\delta} \le \frac{\varepsilon}{2}$ for $n \ge n_0$. Finally, $|\sigma_n(z) - f(z)| \le \varepsilon$ for $n \ge n_0$ and $z \in D$, and therefore for $z \in \overline{D}$. □

3. Mittag-Leffler theorem.

THEOREM 3.1 (Mittag-Leffler theorem, cf. [**5**]). *There exist numbers $c_{n,j} \in \mathbb{C}$, $n \in \mathbb{N}$, $j \in \{0, \dots, k_n\}$, such that for every $N \in \mathbb{N}$, for every starlike domain $D \subset \mathbb{C}^N$, and for every $f \in \mathcal{O}(D)$ with the Taylor development*

$$f(z) = \sum_{j=0}^{\infty} Q_j(z)$$

in a neighborhood of 0*, the sequence of polynomials*

$$\sigma_n := \sum_{j=0}^{k_n} c_{n,j} Q_j, \quad n \in \mathbb{N},$$

converges to f locally uniformly in D. In particular, the sequence $\{\sigma_n\}_{n=1}^{\infty}$ is locally uniformly convergent in the maximal starlike domain G_f to which f is analytically continuable (G_f is called the Mittag-Leffler star*).*

REMARK 3.2. (a) The case $N = 1$ is due to Mittag-Leffler (cf. [**12**]).
(b) Our proof of Theorem 3.1 will be based on a method proposed (for $N = 1$) by E. Borel (cf. [**2**]).
(c) Generalizations of the Mittag-Leffler theorem to the case $N \ge 2$ were studied (using various methods) by several authors (cf. e.g. [**1**], [**10**], [**5**]).
(d) Theorem 3.1 implies that polynomials are dense in $\mathcal{O}(D)$ (see also [**4**]). In particular, the envelope of holomorphy $\widetilde{D}$ of D is a starlike Runge domain (cf. Remark 1.1(c)).

Proof of Theorem 3.1. By Runge's theorem there exists a sequence $\{W_n\}_{n=1}^{\infty}$ of polynomials that converges locally uniformly in $\mathbb{C} \setminus [1, +\infty)$ to the function $W(\lambda) := \frac{1}{1-\lambda}$. Let

$$W_n(\lambda) = \sum_{j=0}^{k_n} c_{n,j}\lambda^j, \quad n \in \mathbb{N}.$$

Fix $f \in \mathcal{O}(D)$ and $a \in D$. Since $[0,1] \cdot a$ is a compact subset of D there exists an $r > 0$ such that $\Delta \cdot \overline{\mathbb{B}}(a,r) \subset D$, where

$$\Delta := \{x + iy \in \mathbb{C} : -r \le x \le 1 + r,\ |y| \le r\}.$$

For every $z \in \mathbb{B}(a,r)$, the function $\lambda \longmapsto f(\lambda z)$ is holomorphic in a neighborhood of Δ. In particular,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda z)d\lambda}{\lambda - 1}, \quad z \in \mathbb{B}(a,r),$$

where Γ is the positively oriented boundary of Δ. On the other hand, for $z \in \mathbb{B}(a,r)$ we get

$$\sigma_n(z) = \sum_{j=0}^{k_n} c_{n,j}\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda z)d\lambda}{\lambda^{j+1}} = \frac{1}{2\pi i}\int_\Gamma f(\lambda z)W_n\Big(\frac{1}{\lambda}\Big)\frac{d\lambda}{\lambda}.$$

Consequently,

$$f(z) - \sigma_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda z)}{\lambda}\Big(\frac{1}{1-\frac{1}{\lambda}} - W_n\Big(\frac{1}{\lambda}\Big)\Big)d\lambda, \quad z \in \mathbb{B}(a,r).$$

Since $W_n(\frac{1}{\lambda}) \longrightarrow \frac{1}{1-\frac{1}{\lambda}}$ uniformly for $\lambda \in \partial\Delta$, we conclude that $\sigma_n \longrightarrow f$ uniformly on $\mathbb{B}(a,r)$. $\square$

4. Proof of Theorem 1.3. The case of the Bergman boundary follows from Remark 1.2(e).

(a) It suffices to show that $\mathcal{A}(D) \subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$ (cf. Remark 1.2(b)). Fix an $f \in \mathcal{A}(D)$ and let $\{\sigma_n\}_{n=1}^\infty$ be as in § 2. Using Theorem 2.1 and the equation (*) of Remark 1.1 we conclude that the sequence $\{\sigma_n\}_{n=1}^\infty$ is uniformly convergent on $\overline{\widetilde{G}}$ to a function $\widetilde{f} \in \mathcal{A}(\widetilde{D})$, which completes the proof of (a).

(b) Fix a sequence $\varepsilon_n \searrow 0$ and let $D_n := (1 + \varepsilon_n)D \supset \overline{D}$, $n \in \mathbb{N}$. Take an $f \in \mathcal{A}(D)$ and let $f_n(z) := f(\frac{z}{1+\varepsilon_n})$, $z \in D_n$. Then $f_n|_{\overline{D}} \in \mathcal{O}(\overline{D})$ and $f_n|_{\overline{D}} \longrightarrow f$ uniformly on $\overline{D}$. Thus $f \in \mathcal{B}(D)$. Now, the result follows from Remark 1.2(e)(f). $\square$

Remark 4.1. Observe that Theorem 1.3(b) may be also proved via Theorem 3.1. In fact, applying Theorem 3.1 to (D_n, f_n), we conclude that for each $n \in \mathbb{N}$, there exists a polynomial P_n such that $|f_n(z) - P_n(z)| \le \frac{1}{n}$, $z \in \overline{D}$. Thus the sequence $\{P_n\}_{n=1}^\infty$ converges to f uniformly on $\overline{D}$. Now, we can finish the proof as in (a).

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